The electromagnetic virtual cloud of the ground-state hydrogen atom-a quantum field theory approach

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Abstract. The properties of the virtual cloud around the hydrogen atom in the ground state are studied with the use of quantum field theory methods. The relativistic expression for the electromagnetic energy density around the atom, with the electron spin taken into account, is obtained. The distribution of the angular momentum contained in the cloud and the self-interaction kernel for the electrons bound in atom are also investigated.

1. Introduction

One of the known and interesting properties of a quantum system is fluctuations giving rise to phenomena that are absent in classical physics. In quantum field theory, for instance, the source interacting with a certain field can surround itself with its quanta, creating in that way a cloud of virtual particles with definite spatial distribution dependent on the features of the source. This source may be, for example, a baryon surrounding itself with a meson cloud, it may also be a charged particle emitting and absorbing photons. Recently much work has been devoted to the description and understanding of the properties of such clouds in various phsyical situations (Compagno *et al* 1983, 1987, Passante *et al* 1985, Passante and Power 1987, Persico and Power 1986, Power and Thirunamachandran 1983, 1984a, b, Henley and Thirring 1962, Peeters and Devreese 1983, Theberge *et al* 1980, 1981). A matter of particular interest was the spatial distribution of the virtual photon cloud of the hydrogen atom (Compagno *et al* 1983, 1987, Passante *et al* 1985, Passante and Power 1987, Persico and Power 1986). The virtual cloud arises even if atom (or a source in general) is in its ground state; responsible for it are those terms in Hamiltonian that admit excitations of the source with simultaneous emission of particles. An atom can, therefore, be spontaneously excited, for instance to the 2p state, emitting a photon. Such a fluctuation leads to the emission of the off-mass-shell photon but if it lasts a sufficiently short time (i.e. the photon is immediately reabsorbed and the atom comes back to the former state) is allowed by the Heisenberg uncertainty principle:

$$\Delta E \Delta t \geq \hbar. \tag{1}$$

Phenomena of that kind are well known and manifest themselves through, for example, van der Waals forces, where virtual photons are exchanged between two atoms. For a baryon mentioned above, this process looks similar. A proton or neutron can emit a pion, becoming simultaneously a higher resonance, for instance Δ, and reabsorb it after a short time.

In the present work we will concentrate on a description of the photon cloud in the ground state of the hydrogen atom. What physical quantity can be chosen to characterize such a cloud? Undoubtedly it has to be a quantity that is at least bilinear in the fields since only then can the quantum nature of the source manifest itself (if we are not interested in radiative corrections which come into play for quantities linear in the fields as well). A suitable quantity for this study is the electromagnetic energy density distribution in the space around the atom: $\langle \frac{1}{2}(E(x)^2 + B(x)^2)\rangle$. This was in fact the basic quantity considered in the quoted works. It was found for the non-relativistic, quantum mechanical atom interacting with the quantized electromagnetic field. The Hamiltonian of that system has the form

$$H = \frac{(p - eA)^2}{2m} + V(r) + \frac{1}{2}\int d^3x(E^2 + B^2) \tag{2}$$

where

$$A(r) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3 2|k|}(a_k^{(\lambda)}e_k^{(\lambda)}\,e^{ikr} + a_k^{(\lambda)\dagger}e_k^{(\lambda)*}\,e^{-ikr}) \tag{3}$$

and $a_k^{(\lambda)\dagger}$ $(a_k^{(\lambda)})$ are creation (annihilation) operators for the photons of momentum k and polarization λ.

What we would like to do, however, and this is the main goal of this work, is to carry out all these calculations entirely in the language of quantum field theory. We have no intention of also constructing in such a way the bound state of proton and electron—we consider from the very beginning relativistic quantum electrodynamics in the external Coulomb potential (the corresponding Lagrangian is written down explicitly in section 2—formula (4)). The use of quantum field theoretical techniques permits us to easily find other quantities characterizing the virtual cloud. It also admits generalizations to various similar problems not dealt with in the present work. What we are looking for, is, therefore, the expression

$$\langle 1|\tfrac{1}{2}(E(r,t)^2 + B(r,t)^2)|1\rangle$$

where $|1\rangle$ is the physical ground state of the *system*. If we make use of the eigenbasis of the Hamiltonian in which the electron-photon interaction term was turned off, the state $|1\rangle$ would be decomposed onto free atomic states with various numbers of photons, as given by the ordinary perturbation calculation. This is actually the case dealt with in works based on the Hamiltonian (2) (Compagno *et al* 1983, Passante *et al* 1985, Passante and Power 1987). For us, however, this decomposition is of little avail—we prefer rather to employ the perturbation methods of field theory, leading to Green functions and Feynman diagrams. It will be shown that the average value that we are interested in can actually be found with the use of scattering methods. It is very convenient to transfer our problem into the most elaborated domain in quantum field theory. We get ordinary transition amplitudes and Green functions, which can be dealt with by well known techniques. All this constitutes the contents of section 2. In section 3 we find another quantity characterizing the cloud—the angular momentum density distribution: $\langle r\times(E\times B)\rangle$. Fortunately, the field theoretical method we have applied is universal enough to allow us to obtain, with relatively little expenditure of work, results for all quantities bilinear in the potentials A once the first one has been found. In section 4 we calculate the full energy contained in the virtual cloud and find the self-interaction kernel for the electron bound as an atom.

2. The electromagnetic energy density distribution

In this section we would like to concentrate on the field theoretic decription of the electromagnetic energy density distribution in the space around the hydrogen atom as a possible characteristic of the virtual cloud. The very fact of the existence of such a cloud for a neutral source, like a hydrogen atom, is closely related to its quantum character and, consequently, to its fluctuations—to the continuous processes of emission and absorption of virtual photons. In the system consisting of the atom and the electromagnetic field, as soon as the interaction between the electron and the field A is turned on, a new ground state appears. It is no longer the ground state of the atom alone, which written in the unperturbed product basis would have the form $|1s\rangle \otimes |0\rangle$, but the ground state of the whole coupled system, which mixes the electron and photon degrees of freedom and which in the above basis assumes the form of an infinite series. For such a ground state the expectation values of operators being certain combinations of fields A, will in general not vanish in the space around the atom. We mean here, naturally, the operators that are already normally ordered—after the subtraction of the vacuum contribution, which would give non-zero values even for empty space. Before passing onto concrete calculations of such a quantity—$\langle \frac{1}{2}(E(x)^2 + B(x)^2) \rangle$—we will write down the Lagrangian of the theory we will deal with here. It is the standard Lagrangian of relativistic quantum electrodynamics in the presence of an external Coulomb potential:

$$\mathcal{L}(x) = \bar{\Psi}(\mathrm{i}\gamma^\mu \partial_\mu - m - \gamma^0 V(x))\Psi - e\bar{\Psi}\gamma^\mu A_\mu \Psi + \tfrac{1}{2}A^\mu(\partial^\alpha \partial_\alpha g_{\mu\nu} - (1-\lambda)\partial_\mu \partial_\nu)A^\nu \tag{4}$$

where

$$V(r) = -\frac{Z\alpha}{r}. \tag{5}$$

and λ is a gauge-fixing parameter.

We denote by the symbol $|1\rangle$ the one-electron ground state of that system. The quantity we want to find is simply $\langle 1|\frac{1}{2}(E(r, t)^2 + B(r, t)^2)|1\rangle$. The operator $\frac{1}{2}(E(r, t)^2 + B(r, t)^2)$ is, however, singular since it is a product of two field operators in the same spacetime point. Evaluating this expression directly, we would get infinity just because we neglected to subtract the vacuum contribution. Therefore, we first have to split the spacetime arguments of the fields $(x \leftrightarrow y)$, as one usually does in such a case. Next we subtract the vacuum expectation value of the same operator, obtaining in that way a well defined quantity, and finally we put $y = x$. Our basic object will be then:

$$I^{\mu\nu}(x, y) = \langle 1|\mathcal{A}^\mu(x)\mathcal{A}^\nu(y)|1\rangle - \langle \Omega|A^\mu(x)A^\nu(y)|\Omega\rangle \tag{6}$$

where

$$\mathcal{A}^\mu(x) = A_{\mathrm{cl}}^\mu(x) + A^\mu(x) \tag{7}$$

and $A_{\mathrm{cl}}^\mu(x)$ is the classical Coulomb field coming from the proton, whereas $A^\mu(x)$ is the quantum field. The differentiations leading to fields E and B will be performed later.

The splitting of spacetime points is carried out in such a way that $x_0 > y_0$. We may then, without altering anything in our expression (6), insert the chronological ordering operator.

$$\mathcal{A}^\mu(x)\mathcal{A}^\nu(y) \to \mathrm{T}(\mathcal{A}^\mu(x)\mathcal{A}^\nu(y)).$$

If we now recall that the ground state obeys

$$|1\rangle = |1\text{in}\rangle = |1\text{out}\rangle \tag{8}$$

where the asymptotic states

$$|1_{\text{out}}^{\text{in}}\rangle = |1s, 0_{\text{out}}^{\text{in}}\rangle \tag{9}$$

are created from the vacuum with the creation operator for a physical dressed 1s electron $\rightarrow b_{1s}^{+\text{in}}|0, 0\rangle$ (the second 0 stands for the photon vacuum), we arrive at the expression known from the scattering process:

$$I^{\mu\nu}(x, y) = \langle 1s, 0\text{out}|T(\mathscr{A}^{\mu}(x)\mathscr{A}^{\nu}(y))|1s, 0\text{in}\rangle - \langle \Omega\text{out}|T(A^{\mu}(x)A^{\nu}(y))|\Omega\text{in}\rangle. \tag{10}$$

We have, then, transformed our problem into that of evaluating the S-matrix element for the scattering of a photon on the ground state hydrogen atom. The subsequent way of proceeding is already clear: with the aid of the reduction formulae (Bjorken and Drell 1965, Itzykson and Zuber 1978), one has to form the vacuum expectation values that are the appropriate Green functions. If we recall that $\mathscr{A}^{\mu} = A_{\text{cl}}^{\mu} + A^{\mu}$ we see that there are three contributions to $I^{\mu\nu}$. We will start with $I_{\text{I}}^{\mu\nu}$ that corresponds to the most interesting situation, when both fields A in (10) are quantum fields. The determination of the remaining terms $I_{\text{II}}^{\mu\nu}$ and $I_{\text{III}}^{\mu\nu}$ does not present any difficulties:

$$I_{\text{I}}^{\mu\nu}(x, y) = \langle 1s, 0\text{out}|T(A^{\mu}(x)A^{\nu}(y))|1s, 0\text{in}\rangle - \langle \Omega\text{out}|T(A^{\mu}(x)A^{\nu}(y))|\Omega\text{in}\rangle. \tag{11}$$

The 'zero point' term has been included in $I_{\text{I}}^{\mu\nu}$ as this is the only contribtuion that can lead to the problems with ill defined operators; here we have the product of two quantum fields. Now we will make use of

$$b_{1}^{+\text{in}} = \int \mathrm{d}^3x \, \bar{\Psi}_{\text{in}}(x)\gamma^0\Psi_{1}^{(+)}(x) \tag{12}$$

$$b_{1}^{\text{out}} = \int \mathrm{d}^3x \, \bar{\Psi}_{1}^{(+)}(x)\gamma^0\Psi_{\text{out}}(x) \tag{13}$$

where Ψ_{1} is a suitable ground state wavefunction and Ψ_{in} (Ψ_{out}) are the electron field operators. For our purpose—the calculations in the lowest order of perturbation theory—it will turn out quite sufficient to use as Ψ_{1} the standard wavefunction of the hydrogen atom from quantum mechanics. Now, acting as one usually does while deriving the reduction formulae (Bjorken and Drell 1965; Itzykson and Zuber 1978) we obtain, at first, the disconnected part $\langle 0\text{out}|T(A^{\mu}(x)A^{\nu}(y))|0\text{in}\rangle$ which approximately—to order e^2—cancels the zero-point contribution. The cancellation is not exact (there remains some $O(e^4)$ part) because $|0\rangle$ is not the same as $|\Omega\rangle$, the former being the vacuum state in the presence of external Coulomb potential. The diagram $\sim\!\!\!\bigcirc\!\!\!\sim$, for instance, is different in the two cases, since the electron propagators in the internal loop are different. However, all these corrections come into play at order e^4, which we we will not be interested in.

For the connected part, essential for us, we obtain

$$I_{\text{I}}^{\mu\nu}(x, y) = -\int \mathrm{d}^4w \, \mathrm{d}^4z \, \bar{\Psi}_{1}(w)(\slashed{\partial}_{w} + \mathrm{i}(m + \gamma^0 V))$$

$$\times \langle 0\text{out}|T(\Psi(w)\bar{\Psi}(z)A^{\mu}(x)A^{\nu}(y))|0\text{in}\rangle(\overleftarrow{\slashed{\partial}}_{z} - \mathrm{i}(m + \gamma^0 V))\Psi_{1}(z). \tag{14}$$

The arrow over the derivative $(\overleftarrow{\slashed{\partial}}_{z})$ means that it acts to the left.
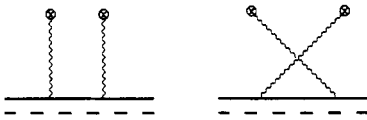
Figure 1. Feynman diagrams that contribute to (14). The full line stands for electron and the broken one for the proton.
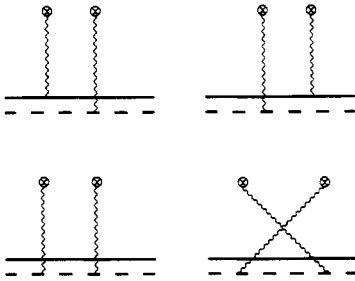

Figure 2. The diagrams contributing to $I_{11}^{\mu\nu}$ and $I_{111}^{\mu\nu}$.

Ordinary perturbation calculation now leads to two Feynman diagrams, the same as for the scattering process. They are shown in figure 1.

In principle we do not follow the quantum calculation for the proton; since there are, however, Coulomb photons coming from the nucleus (figure 2) we decided, for clarity, to draw the proton line too:

$$I_1^{\mu\nu}(x, y) = \mathrm{i}e^2 \int \mathrm{d}^4 w \, \mathrm{d}^4 z \, \bar{\Psi}_1(w) \gamma^\mu S_v^{\mathrm{F}}(w, z) \gamma^\nu \Psi_1(z) \Delta^{\mathrm{F}}(x - w) \Delta^{\mathrm{F}}(y - z) + \begin{Bmatrix} \mu \leftrightarrow \nu \\ x \leftrightarrow y \end{Bmatrix}. \quad (15)$$

S_v^{F} is the Feynman electron propagator in the Coulomb potential V:

$$(\mathrm{i}\slashed{\partial}_x - m - \gamma^0 V(x)) S_v^{\mathrm{F}}(x, w) = \delta^{(4)}(x - w). \quad (16)$$

The photon propagator is taken here in the Feynman gauge ($\lambda = 1$), chosen for reasons of simplicity. It is then expressed in the form

$$D_{\mu\nu}^{\mathrm{F}}(x) = g_{\mu\nu} \Delta^{\mathrm{F}}(x) \quad (17)$$

where $\Delta^F(x)$ has the following coordinate space representation:

$$\Delta^{\mathrm{F}}(x) = \frac{-\mathrm{i}}{4\pi^2} \frac{1}{x^2 - \mathrm{i}\varepsilon}. \quad (18)$$

We would like to emphasize here that in the course of our calculations there will be no need to pass on to the Fourier representation; all the work is done in coordinate space with $\Delta^F(x)$ defined above. We do not know the explicit form for the propagator $S_v^{\mathrm{F}}(w, z)$, but one can always use the representation in the form of a sum over the appropriate atomic states:

$$S_v^{\mathrm{F}}(w, z) = -\mathrm{i} \sum_{n^+}\!\!\int \Theta(w_0 - z_0) \Psi_n^{(+)}(w) \bar{\Psi}_n^{(+)}(z) + \mathrm{i} \sum_{n^-}\!\!\int \Theta(z_0 - w_0) \Psi_n^{(-)}(w) \bar{\Psi}_n^{(-)}(z) \quad (19)$$

where $+$ and $-$ signify positive and negative energy states. As a consequence of this

we obtain

$$
I_1^{\mu\nu}(x, y) = I_{1^+}^{\mu\nu}(x, y) + I_{1^-}^{\mu\nu}(x, y)
$$

$$
= e^2 \int d^4w \, d^4z \sum_{n^+}\!\!\int \Theta(w_0 - z_0) \bar{\Psi}_1(w) \gamma^\mu \Psi_n^{(+)}(w) \bar{\Psi}_n^{(+)}(z) \gamma^\nu \Psi_1(z)
$$

$$
\times \Delta^F(x - w)\Delta^F(y - z)
$$

$$
- e^2 \int d^4w \, d^4z \sum_{n^-}\!\!\int \Theta(z_0 - w_0) \bar{\Psi}_1(w) \gamma^\mu \Psi_n^{(-)}(w) \bar{\Psi}_n^{(-)}(z) \gamma^\nu \Psi_1(z)
$$

$$
\times \Delta^F(x - w)\Delta^F(y - z) + \begin{Bmatrix} \mu \leftrightarrow \nu \\ x \leftrightarrow y \end{Bmatrix}. \tag{20}
$$

The symbol $\sum\!\int$ in the above formulae signifies the sum over discrete and integral over continuous quantum numbers. What remains now is to perform the integrals over w_0 and z_0. As the time dependence of the functions Ψ_n is known—they are stationary solutions of the atomic Dirac equation—these integrals are simple contour integrals with poles situated as is required by the Feynman propagators Δ^F. Leaving unexecuted the integrals over the spatial distribution of the source, we get

$$
I_{1^\pm}^{\mu\nu}(x, y) = \pm e^2 \frac{1}{16\pi^3} \sum_{n^\pm}\!\!\int
$$

$$
\times \int \frac{d^3w \, d^3z}{|x - w||y - z|} \bar{\Psi}_1(w) \gamma^\mu \Psi_n^\pm(w) \bar{\Psi}_n^\pm(z) \gamma^\nu \Psi_1(z) \int_0^\infty dk \frac{1}{k + |E_{n^\pm} - E_1|}
$$

$$
\times \exp[ik(x_0 - y_0)] \sin[k(|x - w| + |y - z|)] + \begin{Bmatrix} \mu \leftrightarrow \nu \\ x \leftrightarrow y \end{Bmatrix}. \tag{21}
$$

The integral over k comes from the Fourier representation of the Heaviside step function $\Theta(x)$:

$$
\Theta(x) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dk \frac{e^{ikx}}{k - i\varepsilon}. \tag{22}
$$

At this point we would like to stress the simplicity and the clarity of this method. The formula (21) is the most complicated one that arises in the course of the calculations.

The results for parts $I_{II}^{\mu\nu}$ and $I_{III}^{\mu\nu}$, which correspond to the diagrams of figure 2, put aside so far, are given here without any calculation:

$$
I_{II}^{\mu\nu}(x, y) = -\frac{e^2}{16\pi^2} \frac{g^{\mu 0} \int d^3z \, \bar{\Psi}_1(z) \gamma^\nu \Psi_1(z)}{|x||y - z|} + \begin{Bmatrix} \mu \leftrightarrow \nu \\ x \leftrightarrow y \end{Bmatrix} \tag{23}
$$

$$
I_{III}^{\mu\nu}(x, y) = \frac{e^2}{16\pi^2} \frac{g^{\mu 0} g^{\nu 0}}{xy}. \tag{24}
$$

In this way we have obtained all the parts of our expression:

$$
I^{\mu\nu} = I_{1^+}^{\mu\nu} + I_{1^-}^{\mu\nu} + I_{II}^{\mu\nu} + I_{III}^{\mu\nu}. \tag{25}
$$

To proceed with (25) we must now make some approximations. We would like to separate x from w and y from z in the formula for $I_1^{\mu\nu}$. Ψ_1 is a localized state, so the integrations over w and z are spread only over the region of order a_0—the Bohr radius. From now on we will assume that the observation point x (and y) is far from the atom.

So we have $|x| \gg a_0 (\approx w)$ and $|y| \gg a_0 (\approx z)$. We do not claim to find the energy density inside the atom. At the same time we always have to keep in mind that there is also the relation that the wavelengths of characteristic photons are much greater than the size of the source: $a_0 \ll \lambda$. We then make the multipolar expansion. This is a lengthy but elementary calculation which we skip here. It consists simply in the Taylor expansion of (21) with respect to w and z. The terms contain two small parameters: a_0/r (i.e. w/x and z/y) and a_0/λ_{n1}. We reject all higher (non-dipole transitions and take the leading term in the a_0/λ_{n1} expansion. This is the usual approach when investigating, for example, multipolar radiation. We are left then with the objects $\langle 1|r^i|n \rangle \langle n|r^j|1 \rangle$. From now on we use, for brevity, the symbol $|n\rangle$ to denote the quantum mechanical atomic states Ψ_n. We hope it will not cause confusion with field theory states, for instance $|1\rangle$.

After the above approximation has been made we have to calculate the derivatives leading to the electric and magnetic fields: $\partial^i A^0 \partial^i A^0$, $\partial^i A^0 \partial^0 A^i$, $\partial^0 A^i \partial^0 A^i$ for $\boldsymbol{E}^2$ and $(\mathrm{curl}\, \boldsymbol{A})^2$ for $\boldsymbol{B}^2$ and then to evaluate the k integral. This gives the integral sine and the integral cosine functions. In that manner we obtain the following formula for $\boldsymbol{E}^2$:

$$\tfrac{1}{2}\langle \boldsymbol{E}^2 \rangle = \frac{e^2}{32\pi^3} \sum_n \int' \langle 1|x^k|n \rangle \langle n|x^i|1 \rangle$$

$$\times \left[\left(\frac{2}{r^6}(\delta^{ik} + 3\hat{r}^k\hat{r}^i) - \frac{2\omega_{n1}^2}{r^4}(3\delta^{ik} + \hat{r}^k\hat{r}^i) + \frac{2\omega_{n1}^4}{r^2}(\delta^{ik} - \hat{r}^k\hat{r}^i) \right) f(2\omega_{n1}r) \right.$$

$$\left. + \left(\frac{4\omega_{n1}}{r^5}(3\delta^{ik} + \hat{r}^k\hat{r}^i) - \frac{4\omega_{n1}^3}{r^3}(\delta^{ik} - \hat{r}^k\hat{r}^i) \right) g(2\omega_{n1}r) - \frac{\omega_{n1}^3}{r^3}(\delta^{ik} - \hat{r}^k\hat{r}^i) \right]$$

$$(26)$$

where

$$\sum_n \int' = \sum_{n^+} \int - \sum_{n^-} \int \qquad n = 1 \text{ excluded}$$

$$\omega_{n1} = |E_n - E_1| = \begin{cases} E_n - E_1 & \text{for positive-energy states} \\ E_1 - E_n & \text{for negative-energy states.} \end{cases}$$

The well known (Abramowitz and Stegun 1964) functions $f(z)$ and $g(z)$ are expressed as follows:

$$f(z) = \mathrm{ci}(z)\sin(z) - \mathrm{si}(z)\cos(z) \qquad\qquad (27)$$

$$g(z) = -\mathrm{ci}(z)\cos(z) - \mathrm{si}(z)\sin(z) \qquad\qquad (28)$$

where we use the definitions

$$\mathrm{si}(z) = \int_\infty^z \frac{\sin(t)}{t}\, \mathrm{d}t \qquad\qquad (29)$$

$$\mathrm{ci}(z) = \int_\infty^z \frac{\cos(t)}{t}\, \mathrm{d}t. \qquad\qquad (30)$$

In (26), $I_{11}^{\mu\nu}$ and $I_{111}^{\mu\nu}$ are already taken into account. They have cancelled the monopole term of our multipolar expansion, as it should be for a neutral object, which the hydrogen atom is. In the same manner we get the magnetic energy density:

$$\tfrac{1}{2}\langle \boldsymbol{B}^2 \rangle = \frac{e^2}{16\pi^3} \sum_n \int' \langle 1|x^k|n \rangle \langle n|x^i|1 \rangle \left\{ \left(\frac{\omega_{n1}^2}{r^4} - \frac{\omega_{n1}^4}{r^2} \right) f(2\omega_{n1}r) + \frac{2\omega_{n1}^3}{r^3} g(2\omega_{n1}r) + \frac{\omega_{n1}^3}{2r^3} \right\}$$

$$\times (\delta^{ik} - \hat{r}^k\hat{r}^i) + \frac{e^2}{128m^2}\left(\frac{1+2\gamma}{3} \right)^2 (1 + 3(\hat{x}\hat{s})^2)\frac{1}{r^6}. \qquad (31)$$

The last term of (31) is a magnetostatic energy coming from the spin magnetic moment of the electron. For the total energy density contained in the cloud we obtain:

$$\frac{1}{2}\langle E^2 + B^2 \rangle = \frac{e^2}{16\pi^3} \sum_n \int' \langle 1|x^k|n\rangle\langle n|x^i|1\rangle \left[\left(\frac{1}{r^6}(\delta^{ik}+3\hat{r}^k\hat{r}^i) - \frac{2\omega_{n1}^4}{r^2}(\delta^{ik}+\hat{r}^k\hat{r}^i) \right) \right.$$

$$\left. \times f(2\omega_{n1}r) + \frac{2\omega_{n1}}{r^5}(\delta^{ik}+3\hat{r}^k\hat{r}^i)g(2\omega_{n1}r) \right] \qquad (32)$$

where we have disregarded the magnetostatic energy as we are interested here only in the virtual cloud effects. We disregard it also in what follows. The formula (32) constitutes an answer to the question asked at the beginning: what is the photon cloud around the hydrogen atom in the ground state like? This formula, however, is not quite legible as one cannot see clearly the r dependence. We would like to separate a quantity that characterizes the atom and to explicitly find the external r dependence. The summation over the atomic states, which would lead to this goal, cannot, however, be performed; one can instead make use of the well known behaviour of the functions f and g for big arguments (Abramowitz and Stegun 1964):

$$f(z) = \frac{1}{z}\left(1 - \frac{2!}{z^2} + \frac{4!}{z^4} - \frac{6!}{z^6} + \ldots \right) \qquad (33)$$

$$g(z) = \frac{1}{z^2}\left(1 - \frac{3!}{z^2} + \frac{5!}{z^4} - \frac{7!}{z^6} + \ldots \right). \qquad (34)$$

Large arguments (i.e. $r \gg 1/\omega_{n1}$) mean that we look at the atom from the so-called wave zone (far zone). We are at distances much larger than the wavelength of characteristic photons. Keeping only leading terms after having inserted the expansions (33) and (34) into the formulae (26), (31) and (32) we have:

$$\frac{1}{2}\langle E^2(r)\rangle_{FZ} = \frac{13e^2}{64\pi^3} \sum_n \int \frac{1}{E_n - E_1} \langle 1|x|n\rangle\langle n|x|1\rangle \frac{1}{r^7}$$

$$+ \frac{7e^2}{64\pi^3} \sum_n \int \frac{1}{E_n - E_1} \langle 1|x^i|n\rangle\langle n|x^k|1\rangle \frac{\hat{r}^i\hat{r}^k}{r^7} \qquad (35)$$

$$\frac{1}{2}\langle B^2(r)\rangle_{FZ} = -\frac{7e^2}{64\pi^3} \sum_n \int \frac{1}{E_n - E_1} \langle 1|x^i|n\rangle\langle n|x^k|1\rangle(\delta^{ik} - \hat{r}^i\hat{r}^k) \frac{1}{r^7} \qquad (36)$$

$$\frac{1}{2}\langle E^2(r) + B^2(r)\rangle_{FZ} = \frac{e^2}{64\pi^3} \sum_n \int \frac{1}{E_n - E_1} \langle 1|x^i|n\rangle\langle n|x^k|1\rangle(3\delta^{ik} + 7\hat{r}^i\hat{r}^k) \frac{1}{r^7}. \qquad (37)$$

The above formulae constitute relativistic, bispinor generalization of results already obtained in the language of non-relativistic QED. (Passante *et al* 1985, Passante and Power 1987, Persico and Power 1986). If we had to do with a spherically symmetric atom we would have

$$\frac{1}{2}\langle E^2(r)\rangle_{FZ}^{SF\,SYM} = \frac{1}{4\pi} \frac{23e^2}{24\pi^2} \sum_n \int \frac{1}{E_n - E_1} \langle 1|x|n\rangle\langle n|x|1\rangle \frac{1}{r^7} \qquad (38)$$

$$\frac{1}{2}\langle B^2(r)\rangle_{FZ}^{SF\,SYM} = -\frac{1}{4\pi} \frac{7e^2}{24\pi^2} \sum_n \int \frac{1}{E_n - E_1} \langle 1|x|n\rangle\langle n|x|1\rangle \frac{1}{r^7}. \qquad (39)$$

Apart from the factor $1/4\pi$ (coming from a different choice of electromagnetic units) and the relativistic character of the $|n\rangle$, these are exactly formulae already obtained in Passante and Power (1987).

What we have found is some additional non-trivial angular dependence related to the presence of spin. Spin changes the spatial shape of the cloud; the surfaces of constant energy density are no longer spherical—they are now spheroids characterized by the equation

$$A(x^2 + y^2 + z^2) + Bz^2 = (x^2 + y^2 + z^2)^{9/2}.$$

In this frame, spin has only a z component. It is the coefficient B that deforms the sphere.

3. The angular momentum density

The quantum field theoretical method we have applied to find the energy density permits us to make some generalizations. In the course of the calculation in section 2 we have found very useful objects $I^{\mu\nu}(x, y)$ which can now constitute a starting point for evaluating another quantity characterizing the virtual cloud—the density of the angular momentum associated with electromagnetic field: $\langle 1|r \times (E(r, t) \times B(r, t))|1\rangle$. The fact that inside the cloud there is energy flux, and angular momentum connected with it, is exclusively a result of the presence of spin, and in the model (2) this quantity is exactly zero. The experience acquired with the energy density calculation permits us now to find $\langle 1|r \times (E(r, t) \times B(r, t))|1\rangle$ without any problems. It requires only a good deal of patience (similarly as for $\langle E^2 + B^2 \rangle$) while acting with a large number of different terms coming from the multipolar expansion and from the differentiations giving E and B. We give, therefore, only the final result:

$$\langle (r \times (E(r) = B(r)))_j \rangle$$

$$= -\frac{ie^2}{16\pi^3} \sum_n \int' \left(\langle 1|x^k|n\rangle\langle n|x^i|1\rangle + \langle 1|x^i|n\rangle\langle n|x^k|1\rangle \right) \left[\omega_{n1} \left(\frac{2}{r^5} - \frac{4\omega_{n1}^2}{r^3} \right) \right.$$

$$\left. \times f(2\omega_{n1}r) + \omega_{n1} \left(\frac{2}{r^4} - \frac{4\omega_{n1}^2}{r^2} \right) g(2\omega_{n1}r) + \frac{5\omega_{n1}}{2r^4} \right] \varepsilon_{jli} \frac{r^l r^k}{r}. \tag{40}$$

In the far zone it takes the form

$$\langle (r \times (E(r) \times B(r)))_j \rangle_{\text{FZ}}$$

$$= \frac{35ie^2}{16\pi^3} \sum_n \int \frac{1}{(E_n - E_1)^2} \left(\langle 1|x^k|n\rangle\langle n|x^i|1\rangle - \langle 1|x^i|n\rangle\langle n|x^k|1\rangle \right) \varepsilon_{jli} \frac{\hat{r}^l \hat{r}^k}{r^7}. \tag{41}$$

The sum occurring above may only have the following structure:

$$\sum_n \int \frac{1}{(E_n - E_1)^2} \langle 1|x^k|n\rangle\langle n|x^i|1\rangle = \alpha\delta^{ik} + \beta\hat{s}^i\hat{s}^k + \gamma\varepsilon^{ikl}\hat{s}^l. \tag{42}$$

The first two terms, symmetric in i and k, cancel in (41) (this is why we would get zero for the spherically symmetric atom) and the only contribution comes from the third term. That means that the angular momentum density can only have a L_θ component, where θ is the angle between the direction of the spin (the z axis) and the observation point r. It assumes its largest value on the equator and vanishes at the poles of the atom. It looks as though the cloud 'rotates' around the spin axis.

4. The electron self-interaction kernel

In this section, starting again from the object $I^{\mu\nu}(x, y)$ (or more exactly $I_1^{\mu\nu}(x, y)$) we will find one more object which gives an insight into the virtual cloud—the total energy stored up. Yet we will not be interested here in a concrete number as a result of the integration, but rather in the structure of the expression without carrying out the integrals over the spatial distribution of the source. We will then have to calculate the integral $\int d^3r \frac{1}{2}\langle 1|E^2(r) + B^2(r)|1\rangle$ over all space—inside the atom as well—which means that now we cannot make use of any approximation, not only that of the wave zone but the multipolar expansion either. We have therefore to handle full expressions of the type (21), which after having executed the integral over k are already pretty complicated. In the calculations in question only self-interaction of the electron is considered ($I_1^{\mu\nu}$). Taking into account of all the terms ($I_{II}^{\mu\nu}$ and $I_{III}^{\mu\nu}$ too) would immediately lead to infinity because of the point-like nature of the proton. The calculations are long and laborious. The simplest way is to first find the integral $\int d^3x \, I^{\mu\nu}(x_0, x; y_0, x)$ without taking derivatives leading to E and B. These may be taken later as derivatives over w and z by virtue of the fact that $I_1^{\mu\nu}(x, y)$ depends on x and y only through combinations $|x - w|$ and $|y - z|$. However, the integral $\int d^3x \, I_1^{\mu\nu}$ is infinite. It becomes well defined only after taking the derivatives over w and z (or x and y). On the other hand, the performing of these derivatives complicates the expression very much. We proceed, therefore, another way. Before moving operators d/dw^i, d/dz^i from under the integral $\int d^3x$, we regularize it by adding a term

$$\exp[-\varepsilon(|x - w| + |y - z|)].$$

Now, without executing the k integral in (21) we calculate

$$I = \int d^3x \frac{\exp[(ik - \varepsilon)(|x - w| + |x - z|)]}{|x - w||x - z|}. \tag{43}$$

We shift the integration variable: $x \to x + w$ and the expression (43) depends now on only one external vector ($\xi = w - z$) which we choose as the direction of the third axis for integration in polar coordinates. Taking the easy integral over ϕ and introducing a new variable $t = \sqrt{x^2 + \xi^2 + 2x\xi \cos \theta}$ (in the place of θ), we finally get

$$I = \frac{-2\pi}{ik - \varepsilon} \exp[(ik - \varepsilon)\xi]. \tag{44}$$

For regularized $I_1^{\mu\nu}$ that means:

$$\int d^3x \, I_1^{\mu\nu}(x_0, x; y_0, x)$$

$$= \frac{e^2}{16\pi^2} \sum_n{}' \int d^3w \, d^3z \, \bar{\Psi}_1(w) \gamma^\mu \Psi_n(w) \bar{\Psi}_n(z) \gamma^\nu \Psi_1(z)$$

$$\times \int_0^\infty dk \frac{\exp[ik(x_0 - y_0)]}{k + \omega_{n1}} \left(\frac{\exp(ik|w - x|)}{k + i\varepsilon} + \frac{\exp(-ik|w - z|)}{k - i\varepsilon} \right)$$

$$\times \exp(-\varepsilon|w - z|) + \{\mu \leftrightarrow \nu, x_0 \leftrightarrow y_0, w \leftrightarrow z\}. \tag{45}$$

Now we can perform the differentiations and remove the regularization parameter ε. Gathering everything together, we come to

$$\int \mathrm{d}^3 x \langle \tfrac{1}{2} \boldsymbol{E}^2(\boldsymbol{x}) \rangle$$

$$= \frac{e^2}{8\pi^2} \sum_n \!{}' \int \mathrm{d}^3 w \, \mathrm{d}^3 z \left[\bar{\Psi}_1(\boldsymbol{w}) \gamma^0 \Psi_n(\boldsymbol{w}) \bar{\Psi}_n(\boldsymbol{z}) \gamma^0 \Psi_1(\boldsymbol{z}) \right.$$

$$\times \left(\frac{2}{|\boldsymbol{w}-\boldsymbol{z}|} f(\omega_{n1}|\boldsymbol{w}-\boldsymbol{z}|) - \omega_{n1} g(\omega_{n1}|\boldsymbol{w}-\boldsymbol{z}|) \right)$$

$$+ \mathrm{i}(\bar{\Psi}_1(\boldsymbol{w}) \gamma^0 \Psi_n(\boldsymbol{w}) \bar{\Psi}_n(\boldsymbol{z}) \gamma^i \Psi_1(\boldsymbol{z}) + \bar{\Psi}_1(\boldsymbol{w}) \gamma^i \Psi_n(\boldsymbol{w}) \bar{\Psi}_n(\boldsymbol{z}) \gamma^0 \Psi_1(\boldsymbol{z}))$$

$$\times \frac{w_i - z_i}{|\boldsymbol{w}-\boldsymbol{z}|} \left(\frac{1}{|\boldsymbol{w}-\boldsymbol{z}|} - \omega_{n1} f(\omega_{n1}|\boldsymbol{w}-\boldsymbol{z}|) \right)$$

$$\left. + \bar{\Psi}_1(\boldsymbol{w}) \gamma^i \Psi_n(\boldsymbol{w}) \bar{\Psi}_n(\boldsymbol{z}) \gamma^i \Psi_n(\boldsymbol{z}) \omega_{n1} g(\omega_{n1}|\boldsymbol{w}-\boldsymbol{z}|) \right] \tag{46}$$

$$\int \mathrm{d}^3 x \langle \tfrac{1}{2} \boldsymbol{B}^2(\boldsymbol{x}) \rangle$$

$$= \frac{e^2}{8\pi^2} \sum_n \!{}' \int \mathrm{d}^3 w \, \mathrm{d}^3 z \, \bar{\Psi}_1(\boldsymbol{w}) \gamma^i \Psi_n(\boldsymbol{w}) \bar{\Psi}_n(\boldsymbol{z}) \gamma^k \Psi_1(\boldsymbol{z})$$

$$\times \left[\left(\delta^{ik} + \frac{(w_i - z_i)(w_k - z_k)}{|\boldsymbol{w}-\boldsymbol{z}|^2} \right) \frac{1}{|\boldsymbol{w}-\boldsymbol{z}|} f(\omega_{n1}|\boldsymbol{w}-\boldsymbol{z}|) \right.$$

$$\left. - \left(\delta^{ik} - \frac{(w_i - z_i)(w_k - z_k)}{|\boldsymbol{w}-\boldsymbol{z}|^2} \right) \omega_{n1} g(\omega_{n1}|\boldsymbol{w}-\boldsymbol{z}|) \right]. \tag{47}$$

The above expressions have, as expected, the form $\iint JKJ$, where the J are the electron transition currents. The self-interaction kernel K is a complicated tensor $(K^{\mu\nu})$ expressed through integral functions; it is, however, very easy to pick up from it the Coulomb interaction which corresponds to the term $n = 1$ in the sum over the atomic states in (46) and the spin magnetic moment interaction in the same way from (47):

$$\int \mathrm{d}^3 x \, \tfrac{1}{2} \langle \boldsymbol{E}^2(\boldsymbol{x}) \rangle_{n=1} = \frac{e^2}{8\pi} \int \frac{\mathrm{d}^3 w \, \mathrm{d}^3 z}{|\boldsymbol{w}-\boldsymbol{z}|} \bar{\Psi}_1(\boldsymbol{w}) \gamma^0 \Psi_n(\boldsymbol{w}) \bar{\Psi}_n(\boldsymbol{z}) \gamma^0 \Psi_1(\boldsymbol{z}) \tag{48}$$

$$\int \mathrm{d}^3 x \, \tfrac{1}{2} \langle \boldsymbol{B}^2(\boldsymbol{x}) \rangle_{n=1}$$

$$= \frac{e^2}{16\pi} \int \frac{\mathrm{d}^3 w \, \mathrm{d}^3 z}{|\boldsymbol{w}-\boldsymbol{z}|} \bar{\Psi}_1(\boldsymbol{w}) \gamma^i \Psi_n(\boldsymbol{w}) \bar{\Psi}_n(\boldsymbol{z}) \gamma^k \Psi_1(\boldsymbol{z})$$

$$\times \left(\delta^{ik} + \frac{(w^i - z^i)(w^k - z^k)}{|\boldsymbol{w}-\boldsymbol{z}|^2} \right) \tag{49}$$

where we have used $f(0) = \pi/2$ (f and g are functions defined already in (27) and

(28)). The formulae (46) and (47) show the complicated nature of the electron self-interaction even in the lowest order of perturbation expansion. The energy contained in the cloud is one of the important contributions to the Lamb shift.

5. Summary

In this work we have applied quantum field theoretical methods to the investigation of the virtual photon cloud surrounding a hydrogen atom in the ground state. This was possible thanks to reducing the problem of evaluating the average values $\frac{1}{2}\langle E^2 \rangle$ etc to the calculation of the transition amplitudes known from scattering theory. The formulae we have obtained in section 2 constitute a relativistic, bispinor generalization of those obtained earlier in the language of non-relativistic QED (Passante *et al* 1985, Passante and Power 1987, Persico and Power 1986). If we assume spherical symmetry of the atom, our results are identical to those of the quoted works. When we take into account the spin which spoils spherical symmetry, we find some angular distribution of the cloud.

All our work seems to be much simpler than that of previous studies. The applied method of calculation has allowed us additionally to calculate the density of the angular momentum contained in the cloud. Both quantities, the energy density and the angular momentum density, have in the far zone the behaviour $1/r^7$, which remains in agreement with calculations of van der Waals forces with retardation effects taken into account (Aub *et al* 1957, Casimir and Polder 1948, Power and Zienau 1957). In section 4 we have also found the self-interaction kernel for the electron bound in an atom. It is expressed through integral sine and cosine functions (contained in f and g).

The field theoretical approach we have developed is, at least in principle, generalizable to other problems where virtual clouds come into play. In a forthcoming paper we will consider the application of field theoretic methods to the virtual cloud around an atom in an unstable state.

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